Vector Algebra

Let $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

1. $\mathbf{u} + \mathbf{v} = (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j} + (u_3 + v_3)\mathbf{k} = (u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3)$
2. $a\mathbf{u} = (au_1)\mathbf{i} + (au_2)\mathbf{j} + (au_3)\mathbf{k} = (au_1,\ au_2,\ au_3)$
3. $||\mathbf{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$
4. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative law)
5. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associative law)
6. $(ab)\mathbf{u} = a(b\mathbf{u})$ (Associative law)
7. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ (Distributive law)
8. $(a + b)\mathbf{u} =$au + bu (Distributive law)
9. $||a\mathbf{u}|| = |a|\,||\mathbf{u}||$
10. $\left|\left|\frac{\mathbf{v}}{||\mathbf{v}||}\right|\right| = 1$ (The vector $\frac{\mathbf{v}}{||\mathbf{v}||}$ is called the *unit vector in the direction of* $\mathbf{v}$.)
11. $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ (Algebraic definition of the **dot product**.)
12. $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||\cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
13. $\mathbf{u} \cdot \mathbf{v} = 0$ means that $\mathbf{u}$ and $\mathbf{v}$ are perpendiccular.
14. $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$ (another, less usefull, way to say this is $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$)
15. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (Commutative law)
16. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (Distributive law)
17. $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$ (Associative law)
18. $\mathbf{proj_u}\,\mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||^2}\right)\mathbf{u} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||}\right)\frac{\mathbf{u}}{||\mathbf{u}||} = \left(\mathbf{v} \cdot \frac{\mathbf{u}}{||\mathbf{u}||}\right)\frac{\mathbf{u}}{||\mathbf{u}||}.$
19. $\text{comp}_{\mathbf{u}}\,\mathbf{v} = \mathbf{v} \cdot \frac{\mathbf{u}}{||\mathbf{u}||} = ||\mathbf{v}||\cos\theta$ is called the *component of* $\mathbf{v}$ *in the direction of* $\mathbf{u}$.
20. $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ is the equation of the plane containing the given point $\mathbf{p} = (x_1, y_1, z_1)$ and the given normal vector $\mathbf{n} = (A, B, C)$. Another way to write this is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

or even

$$Ax + By + Cz = D \quad \text{where } D = Ax_1 + By_1 + Cz_1.$$

21. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ (Algebraic definition of the **cross product**.)

22. $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||\,||\mathbf{v}|| \sin\theta =$ the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

23. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$ (The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.)

24. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ means that $\mathbf{u}$ and $\mathbf{v}$ are parallel.

25. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ (Anticommutativity)

26. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ (Left distributive law)

27. $(a\mathbf{u}) \times \mathbf{v} = a(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (a\mathbf{v})$ (Associative law)

28. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

29. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ (Notice that $\mathbf{0}$ is the zero *vector*, not the number zero.)

30. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ (The "scalar triple product")

31. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) =$ the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$

32. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

33. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (The "vector triple product")

34. The vector triple product is often written the following way,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

and it is remembered by using the mnemonic "bac-cab".

35. **NOTE:** The cross product is **not** associative. That is,

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is **not** equal to $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

36. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$