1. Given a point (x_0, y_0), the graph of $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ has many different slopes, or "steepnesses." In fact, the graph will have a different slope in every different direction. The slope of the graph at a given point and in a given direction is called a **directional derivative**.

2. We use the gradient vector to compute the directional derivative of $f(x, y)$ at a given point (x_0, y_0) in a given direction $\mathbf{u}$ (where $\mathbf{u}$ must be a unit vector, $||\mathbf{u}|| = 1$),

$$D_{\mathbf{u}} f(x_0, y_0) = \mathbf{u} \cdot \nabla f(x_0, y_0).$$

   Notice how we are using unit vectors to describe direction.

3. Here is a way to "derive" the above formula for the directional derivative. Start with the equation for the tangent plane.

$$\begin{aligned}
T(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\
&= f(x_0,y_0) + (f_x(x_0,y_0),\, f_y(x_0,y_0)) \cdot (x-x_0,\, y-y_0) \\
&= f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0}) \\
&= f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot \left(\frac{\mathbf{x} - \mathbf{x_0}}{||\mathbf{x} - \mathbf{x_0}||}\, ||\mathbf{x} - \mathbf{x_0}|| \right) \\
&= f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot (\mathbf{u}\, ||\mathbf{x} - \mathbf{x_0}||) \quad (\mathbf{u} = \text{unit vector in the direction of } \mathbf{x} - \mathbf{x_0}) \\
&= f(\mathbf{x_0}) + (\mathbf{u} \cdot \nabla f(\mathbf{x_0}))\, ||\mathbf{x} - \mathbf{x_0}|| \quad (\text{algebraic properties of the dot product}) \\
&= f(\mathbf{x_0}) + (D_{\mathbf{u}} f(\mathbf{x_0}))\, ||\mathbf{x} - \mathbf{x_0}|| \quad (\text{definition of the directional derivative}) \\
&= f(x_0,y_0) + (D_{\mathbf{u}} f(x_0,y_0))\, ||(x-x_0,\, y-y_0)||
\end{aligned}$$

4. Given a point (x_0, y_0), if we compute $c = f(x_0, y_0)$, then the gradient vector $\nabla f(x_0, y_0)$ is perpendicular to the level curve of height c. (Remember that level curves and the gradient vector are two-dimensional objects; they live in the two-dimensional xy-plane, not in the three-dimensional xyz-space.)

5. Given a point (x_0, y_0), the graph of $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ has many different slopes, or "steepnesses." The maximal slope of the graph of $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is in the direction of the gradient vector. Another way to put this is that the maximal directional derivative is in the direction of the gradient. Also, the maximal slope is equal to the length of the gradient,

$$D_{\mathbf{u}_{\nabla f}} f(x_0, y_0) = ||\nabla f(x_0, y_0)||$$

   where $\mathbf{u}_{\nabla f}$ is the unit vector in the direction of the gradient.