This reviews the material up to Section 14.4.

1. Let $\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$. We think of the symbol ∇ as representing a certain kind of vector, but it is not really a vector (it is a very convenient notation for a "differential operator").

2. For a scalar field $f(x, y, z)$ and a vector field $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$,

$$\text{grad}(f) = \nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$

$$\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = M_x + N_y + P_z$$

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}.$$

3. Notice that ∇f can be thought of as a vector ∇ times a scalar f which results in a vector (the gradient vector field). And $\nabla \cdot \mathbf{F}$ can be thought of as a vector dotted with a vector, so the result should be a scalar (the divergence scalar field). And $\nabla \times \mathbf{F}$ can be thought of as the cross product of two vectors, so the result should be a vector (the curl vector field).

4. If a vector field is only 2-dimensional, $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, then

$$\text{curl}(\mathbf{F}) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

5. The **line integral of a scalar field over a curve** C is defined as

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\, \sqrt{x'(t)^2 + y'(t)^2}\; dt$$

where the curve C is parameterized by $x(t)$ and $y(t)$ with $a \le t \le b$.

6. The **line integral of a vector field over a curve** C is defined as

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C M(x, y)\, dx + N(x, y)\, dy = \int_a^b M(x(t), y(t))\, x'(t) + N(x(t), y(t))\, y'(t)\; dt$$

where the curve C is parameterized by $x(t)$ and $y(t)$ with $a \le t \le b$.

7. You can parameterize the graph of a function $y = f(x)$ for $a \le x \le b$ with

$$x(t) = t \quad \text{and} \quad y(t) = f(t) \quad \text{with } a \le t \le b.$$

In particular, you can use this with the graph of any line $y = mx + b$.

8. If $\mathbf{p}_0$ and $\mathbf{p}_1$ are two points, then you can parameterize the line segment that connects $\mathbf{p}_0$ to $\mathbf{p}_1$ with

$$\mathbf{r}(t) = (1-t)\,\mathbf{p}_0 + t\,\mathbf{p}_1 \quad \text{with } 0 \le t \le 1.$$

In particular, if $\mathbf{p}_0 = (x_0, y_0, z_0)$ and $\mathbf{p}_1 = (x_1, y_1, z_1)$, then

$$x(t) = (1-t)x_0 + tx_1, \quad y(t) = (1-t)y_0 + ty_1, \quad \text{and} \quad z(t) = (1-t)z_0 + tz_1.$$

9. You can parameterize a circle (or part of a circle) with radius r using

$$x(t) = r\cos(t) \quad \text{and} \quad y(t) = r\sin(t).$$

10. The line integral of a scalar function $f(x, y)$ over a curve C in the xy-plane can be interpreted as the (signed) area of a vertical "curtain" stretching from C up to the graph of f.

If the scalar function is $f(x, y) = 1$, then the scalar line integral of f over a curve C is the arc length of C.

If the scalar function $f(x, y)$ is a density function (usually written as $\delta(x, y)$), then the scalar line integral of $\delta(x, y)$ over a curve C represents the total mass of a wire shaped like C with the variable density at each point in the wire being given by δ.

11. If a vector field $\mathbf{F}$ is interpreted as a force field, then the vector line integral of $\mathbf{F}$ over a curve C represents the work done to move a particle along the curve C from one end of the curve to the other end. To see how this interpretation works, we divide C up into many small pieces of length ds, where ds is small enough so that we can think of each piece of C as being a straight line segment. The work done when moving a particle along one line segment is, as always, $force \times distance$. The distance here is ds, the length of the line segment, and the force needs to be the *component of* $\mathbf{F}$ *that is in the direction of the line segment* (you should draw yourself a small picture). The unit tangent vector $\mathbf{T}$ points in the same direction as the line segment, because the line segment is supposed to be small enough so that it is straight. (Remember, the unit tangent vector $\mathbf{T}$ is the velocity vector divided by its length, $\mathbf{T} = \mathbf{v}/||\mathbf{v}||$, so the unit tangent vector has length 1.) So the dot product $\mathbf{F} \cdot \mathbf{T}$ is exactly what we need, the component of force in the direction that we are moving. So, for each small segment of C, the work done is $force \times distance = (\mathbf{F} \cdot \mathbf{T})\,ds$. Then the integral sign tells us to add up all the small elements or work to get the total work done while moving over the whole curve, $\int_C (\mathbf{F} \cdot \mathbf{T})\,ds$.

12. **Fundamental Theorem of Line Integrals**: If a vector field $\mathbf{F}$ is the gradient of a scalar function f, so $\mathbf{F} = \nabla f$, then for any curve C, with parameterization $x(t)$ and $y(t)$ with $a \leq t \leq b$, we have

$$\int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = f(x(b), y(b)) - f(x(a), y(a)).$$

That is, in order to evaluate the line integral of $\mathbf{F}$ over C, all we need to do is evaluate the change in the scalar function f at the two end points of C. Such a scalar function f is called a **potential function** for the vector field $\mathbf{F}$ (it is sometimes called a **stem function** for $\mathbf{F}$).

13. The Fundamental Theorem of Line Integrals is analogous to the Fundamental Theorem of Calculus because they both say that to evaluate the integral of a function, you only need to find the change in something called an antiderivative (an antiderivative is a function whose derivative is the function you are integrating). In the case of the Fundamental Theorem of Line Integrals, the scalar function f is an "antiderivative" for the vector field $\mathbf{F}$ since we "differentiate" f to get $\mathbf{F}$, $\nabla f = \mathbf{F}$.

    The biggest, most important, difference between the two fundamental theorems is that for the Fundamental Theorem of Calculus, if the integrand is a continuous function, then an antiderivative function always exits, but for the Fundamental Theorem of Line Integrals, it is possible that no "antiderivative" function (i.e., potential function) exists for a vector field $\mathbf{F}$.

14. If a vector field $\mathbf{F}$ has a potential function f, so $\nabla f = \mathbf{F}$, then $\text{curl}(\mathbf{F}) = 0$. But $\text{curl}(\mathbf{F}) = 0$ does **not** mean that $\mathbf{F}$ has a potential function (see below).

15. It is helpful to make the following definitions.

    - A vector field $\mathbf{F}$ is a **conservative vector field** if it is the gradient of a scalar function, that is, there is a scalar function f such that

    $$\mathbf{F} = \nabla f.$$

    - A vector field $\mathbf{F}$ is **independent of path** if, given any two curves C_1 and C_2 that have the same starting point and the same ending point, then

    $$\int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds.$$

    - If C is a closed curve, so it starts and ends at the same point, then the line integral of a vector field $\mathbf{F}$ over the closed curve C is called the **circulation** of $\mathbf{F}$ around C.

16. Let $\mathbf{F}$ be a vector field defined on a region R. Then the following three statements are equivalent, meaning that if one of them is a true statement about $\mathbf{F}$, then the other two must also be true statements about $\mathbf{F}$

- $\mathbf{F}$ is a conservative vector field.
- $\mathbf{F}$ is independent of path.
- The circulation of $\mathbf{F}$ around every closed curve is zero.

If, in addition, the region R does not have any holes in it (we say the region is **simply connected**), then we can add a fourth equivalent condition to the above list.

- $\text{curl}(\mathbf{F}) = 0$.

17. When a 2-dimensional vector field $\mathbf{F}(x, y)$ is define on a simply connected region and the vector field has the property that $\text{curl}(\mathbf{F}(x, y)) = 0$, then there is a procedure that lets us calculate the potential function $f(x, y)$ for $\mathbf{F}(x, y)$.

    Let $\mathbf{F}(x, y) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ be a vector field with the property

    $$\text{curl}(\mathbf{F}(x, y)) = \frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y} = 0.$$

    We want to find a function $f(x, y)$ that solves $\nabla f(x, y) = \mathbf{F}(x, y)$. More specifically, given the two functions $M(x, y)$ and $N(x, y)$, we want to find $f(x, y)$ such that

    $$\frac{\partial f}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y).$$

    Step 1: Compute the antiderivative of $M(x, y)$ with respect to x. This should give us, according to the equation just above on the left, $f(x, y)$ plus some unknown constant of integration. But that unknown "constant of integration" is in fact an unknown function of y (since the derivative with respect to x of any function of y must be 0). So Step 1 gives us

    $$f(x, y) = \int M(x, y)\,dx + g(y)$$

    where we do not, yet, know what the function $g(y)$ is.

    Step 2: Differentiate the answer from Step 1 with respect to y and set the result equal to $N(x, y)$ (since it must me that $f_y(x, y) = N(x, y)$).

    $$N(x, y) = \frac{\partial}{\partial y}\left[\int M(x, y)\,dx + g(y)\right] = \frac{\partial}{\partial y}\left[\int M(x, y)\,dx\right] + g'(y)$$

    Step 3: Solve the equation from the last step for $g'(y)$.

    Step 4: Find the antiderivative with respect to y of $g'(y)$,

    $$g(y) = \int g'(y)\,dy.$$

    Step 5: Substitute the function $g(y)$ found in Step 4 into the answer from Step 1.

18. The formula from Green's Theorem:

$$\oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$