APPENDIX 2. Some Mathematics for Computer Graphics.

"Mathematics, rightly viewed, possess not only truth,
but supreme beauty - a beauty cold and austere, like that of sculpture."
Bertrand Russell

This appendix draws together and summarizes various mathematical results that are referred to throughout the book. In some cases a brief derivation of a result is given, but this material is mainly for convenient reference.

A2.1 Some Key Definitions for Matrices and their Operations.

In this appendix, we review some fundamental concepts of matrices and ways to manipulate them. More general treatments are available in many books (for instance, [Birk65], [Faux79].

A **matrix** is a rectangular array of elements. The elements are most commonly numbers. A matrix with m rows and n columns is said to be an **m by n matrix**. As an example,

$$A = \begin{pmatrix} 3 & 2 & -5 \\ -1 & 8 & 0 \\ 6 & 3 & 9 \\ 1 & 21 & 2 \end{pmatrix} \qquad \text{(A2.1)}$$

is a 4 by 3 matrix of integers and

B = [1.34, - 6.275, 0.0, 81.6]

is a 1 by 4 matrix, also called a "**4-tuple**" or a vector. In common parlance, a 1 by n matrix is a row vector, and an n by 1 matrix is a column vector.

The individual elements of a matrix are conventionally given lowercase symbols and are distinguished by subscripts: The ijth element of matrix B is denoted as b_{ij}. This is the element in the ith row and jth column, so for matrix A above, $a_{32} = 3$.

A matrix is **square** if it has the same number of rows as columns. In graphics we frequently work with 2 by 2, 3 by 3, and 4 by 4 matrices. Two common square matrices are the **zero matrix** and the **identity matrix**. All of the elements of the zero matrix are zero. All are zero for the identity matrix too, except those along the **main diagonal** (those elements a_{ij} for which $i = j$), which have value 1. The 3 by 3 identity matrix is therefore given by

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A2.1.1. Manipulations with Matrices

A matrix B of numbers may be **scaled** by a number s. Each element of B is multiplied by s. The resulting matrix is denoted sB. For A as given in Equation A2.1, for instance,

$$6A = \begin{pmatrix} 18 & 12 & -30 \\ -6 & 48 & 0 \\ 36 & 18 & 54 \\ 6 & 126 & 12 \end{pmatrix}$$

Two matrices C and D having the same number of rows and columns are said to have the same **shape**. They may be added together. The ijth element of the sum $E = C+D$ is simply the sum of the corresponding elements: $e_{ij} = c_{ij} + d_{ij}$. Thus:

$$\begin{pmatrix} 3 & 2 & -5 \\ -1 & 8 & 0 \\ 6 & 3 & 9 \\ 1 & 21 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 5 & -1 \\ 9 & 8 & -3 \\ 2 & 6 & 18 \\ 4 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 7 & -6 \\ 8 & 16 & -3 \\ 8 & 9 & 27 \\ 5 & 23 & 9 \end{pmatrix}$$

Since matrices can be scaled and added, it is meaningful to define **linear combinations** of matrices (of the same shape), such as $2A - 4B$. The following facts about three matrices A, B, and C of the same shape result directly from these definitions:

$$A + B = B + A$$
$$A + (B + C) = (A + B) + C$$
$$(f + g)(A + B) = fA + fB + gA + gB$$

The **transpose** of a matrix M, denoted M^{T}, is formed by interchanging the rows and columns of M: the ijth element of M^{T} is the jith element of M. Thus the transpose of A in Equation A2.1 is

$$A^T = \begin{pmatrix} 3 & -1 & 6 & 1 \\ 2 & 8 & 3 & 21 \\ -5 & 0 & 9 & 2 \end{pmatrix}$$

The transpose of a row vector is a column vector. For example,

$$(3,2,-5)^T = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$$

A matrix is **symmetric** if it is identical to its own transpose. Only square matrices can be symmetric. Thus an n by n matrix M is symmetric if $m_{ij} = m_{ji}$ for i and j between 1 and n.

A2.1.2 Multiplying Two Matrices.

The transformations first discussed in Chapter 5 involve multiplying a vector by a matrix and multiplying two matrices together. The first is a special case of the second.

The **product** AB of two matrices A and B is defined only if the matrices **conform**. That means that the number of columns of the first matrix, A, equals the number of rows of the second one, B. Thus, if A is 3 by 5 and B is 5 by 2, then AB is defined but BA is not. Each term of the product $C = AB$ of A with B is simply the dot product of some row of *A* with some column of *B*. Specifically, the *ij*th element c_{ij} of the product is the dot product of the *i*th row of *A* with the *j*th column of *B*. Thus the product of an n by m matrix with an m by r matrix is an *n* by *r* matrix. For example:

$$\begin{pmatrix} 2 & 0 & 6 & -3 \\ 8 & 1 & -4 & 0 \\ 0 & 5 & 7 & 1 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ -1 & 1 \\ 3 & 1 \\ -5 & 8 \end{pmatrix} = \begin{pmatrix} 45 & -14 \\ 35 & 13 \\ 11 & 20 \end{pmatrix}$$

Here, for instance, $c_{12} = -14$, since (2, 0, 6, - 3) · (2, 1, 1, 8)= - 14. A routine to multiply square matrices is given in Appendix 3. It is easily extended to find the product of any two matrices that conform.

We list some useful properties of matrix multiplication. Assume that matrices *A*, *B*, and *C* conform properly. Then

$$(AB)C = A(BC)$$
$$A(B+C) = AB + AC$$
$$(A+B)C = AC + BC$$
$$(AB)^T = B^T A^T$$
$$A(sB) = sAB$$

where *s* is a number.

When forming a product of two matrices *A* and *B*, the order in which they are taken makes a difference. For the expression *AB*, we say "*A* **premultiplies** *B*" or "*A* is **postmultiplied** by *B*." If *A* and *B* are both square matrices of the same size they conform both ways, so *AB* and *BA* are both well defined, but the two products may contain different elements. If *AB* = *BA* for two matrices, we say that they **commute**. (Do two symmetric matrices always commute?)

Multiplying a Vector by a Matrix.
A special case of matrix multiplication occurs when one of the matrices is a row vector or column vector. In graphics we often see a column4 vector **w** being premultiplied by a matrix, *M*, in the form *M***w**. For example, let

$$w = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} = (2,5,-3)^T$$

and

$$M = \begin{pmatrix} 2 & 0 & 6 \\ 8 & 1 & -4 \\ 0 & 5 & 7 \end{pmatrix}$$

Then **w** conforms with *M*, and we can form

$$M\mathbf{w} = \begin{pmatrix} 2 & 0 & 6 \\ 8 & 1 & -4 \\ 0 & 5 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ 33 \\ 4 \end{pmatrix}$$

By the same rules as those given previously, each component of $M\mathbf{w}$ is the dot product of the appropriate row of M with $\mathbf{w}$. One can also premultiply a matrix by a row vector $\mathbf{v}$, as in

$$\mathbf{v}M = (3,-1,7) \begin{pmatrix} 2 & 0 & 6 \\ 8 & 1 & -4 \\ 0 & 5 & 7 \end{pmatrix} = (-2, 34, 71)$$

The Dot and Cross Products Revisited.
It is useful in some analytical derivations to write the dot product $\mathbf{a} \cdot \mathbf{b}$ of two n-tuples as a vector times a matrix. Simply view vector $\mathbf{b}$ as a row matrix, and transpose it to form the n by 1 column matrix $\mathbf{b}^T$. Then

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}\mathbf{b}^T$$

By the same reasoning, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b}\mathbf{a}^T$.

Similarly, the cross product of two 3-tuples $\mathbf{a} \times \mathbf{b}$ (see Section 4.4) may be written as the product

$$\mathbf{a} \times \mathbf{b} = (a_1, a_2, a_3) \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}$$

The cross product also is some matrix (which one?) postmultiplied by column vector $\mathbf{a}^T$. One other form, the **outer product** or **tensor product** of two vectors, provides a useful notation:

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{a}^T\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1, b_2, b_3) = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}$$

from which $\mathbf{b} \otimes \mathbf{a} = (\mathbf{a} \otimes \mathbf{b})^T$ (why?) An easily proved property is:

$$\mathbf{a}(\mathbf{b} \otimes \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

A2.1.3 Partitioning a Matrix

It is sometimes convenient to subdivide a matrix into blocks of elements and to give names to the various blocks. For example,

$$M = \begin{pmatrix} 2 & 0 & 6 \\ 8 & 1 & -4 \\ 3 & 2 & 7 \end{pmatrix} = \left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right)$$

where the blocks are identified as:

$$M_1 = \begin{pmatrix} 2 & 0 \\ 8 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 6 \\ -4 \end{pmatrix}, M_3 = \begin{pmatrix} 3 & 2 \end{pmatrix}$$

and M_4 consists of the single element 7. This is called a **partition** of M into the four blocks shown. Note that when one block is positioned above another, the two blocks must have the same number of columns. Similarly, when two blocks lie side by side, they must have the same number of rows. Two matrices that have been partitioned in the same way (corresponding blocks have the same shape) may be added by performing these operations on the blocks. To transpose a partitioned matrix, transpose each block individually and then transpose the arrangement of blocks. For instance:

$$\left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right)^{\mathrm{T}} = \left(\begin{array}{c|c} M_1{}^{\mathrm{T}} & M_3{}^{\mathrm{T}} \\ \hline M_2{}^{\mathrm{T}} & M_4{}^{\mathrm{T}} \end{array}\right)$$

You can also multiply two partitioned matrices by multiplying their submatrices in the usual way, as long as the submatrices conform :

$$\left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right)\left(\begin{array}{c|c} M_5 & M_6 \\ \hline M_7 & M_8 \end{array}\right) = \left(\begin{array}{c|c} M_1M_5 + M_2M_7 & M_1M_6 + M_2M_8 \\ \hline M_3M_5 + M_4M_7 & M_3M_6 + M_4M_8 \end{array}\right)$$

A2.1.4 The Determinant of a Matrix

Every square matrix M has a number associated with it called its **determinant** and denoted by $|M|$. The determinant describes the volume of certain geometric shapes and provides information concerning the effect that a linear transformation has on areas and volumes of objects.

For a 2 by 2 matrix M, the determinant is simply the difference of two products:

$$|M| = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21}$$

If M is a 3 by 3 matrix its determinant has the form

$$|M| = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} = m_{11}\begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} - m_{12}\begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} + m_{13}\begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$$

For example:

$$\begin{vmatrix} 2 & 0 & 6 \\ 8 & 1 & -4 \\ 0 & 5 & 7 \end{vmatrix} = 294$$

Note that $|M|$ here is the sum of three terms: $m_{11}M_{11} + m_{12}M_{12} + m_{13}M_{13}$, so it has the form of a dot product: $|M| = (m_{11}, m_{12}, m_{13}) \cdot (M_{11}, M_{12}, M_{13})$. What are the M_{ij} terms? M_{ij} is called the **cofactor** of element m_{ij} for matrix M. We see cofactors emerging again when finding the inverse of a matrix, so it is convenient to define them formally.

Definition Each element m_{ij} of a square matrix M has a corresponding **cofactor** M_{ij}. M_{ij} is $(-1)^{i+j}$ times the determinant of the matrix formed by deleting the ith row and the jth column from M.

Note that as one moves along a row or column, the value of $(-1)^{i+j}$ alternates between 1 and -1. One can visualize a checkerboard pattern of 1's and -1's distributed over the matrix.

The general rule for finding the determinant $|M|$ of any n\yn matrix M is: Pick any row of M, find the cofactor of each element in the row, and take the dot product of the row and the n-tuple of cofactors. Alternatively, pick a column of M and do the same thing. (Does this rule hold for a 2 by 2 matrix as well?)

Some useful properties of determinants are as follows:

- $|M| = |M^T|$
- If two rows (or two columns) of M are identical, $|M| = 0$.
- If M and B are both square, then $|MB| = |M|\,|B|$.
- If B is formed from M by interchanging two rows (or columns) of M, then $|B| = -|M|$.
- If B is formed from M by multiplying one row (or column) of M by a constant k, then $|B| = k\,|M|$.
- If B is formed from M by adding a multiple of one row (or column) of M to another, then $|B| = |M|$.

A2.1.5. The Inverse of a Matrix

An n by n matrix M is said to be **nonsingular** whenever $|M|$• 0. In this case, M has an **inverse**, denoted M^{-1}, that has the property

$$MM^{-1} = M^{-1}M = I$$

where I is the n by n identity matrix. Also, the inverse of a product of square matrices is

$$(AB)^{-1} = B^{-1}A^{-1}$$

It is simple to specify the elements of M^{-1} in terms of cofactors of M:

• Rule for Finding the Inverse of *M*:
Denote the inverse of M by A. Then A has ij-th element

$$a_{ij} = \frac{M_{ji}}{|M|}$$

That is, find the cofactor of the term m_{ji} and divide it by the determinant of the whole matrix. Carefully note the subscripts here: The cofactor of m_{ji} is used when determining a_{ij}. An equivalent procedure is as follows:

1. Build an intermediate matrix C of cofactors: $c_{ij} = M_{ij}$;
2. Find $|M|$ as the dot product of any row of C with the corresponding row of M;
3. Transpose C to get C^T;
4. Scale each element of C^T by $1/|M|$ to form M^{-1}.

Example: Find the inverse of

$$M = \begin{pmatrix} 2 & 0 & 6 \\ 8 & 1 & -4 \\ 0 & 5 & 7 \end{pmatrix}$$

Solution : Build the matrix C of cofactors of M:

$$\begin{pmatrix} 27 & -56 & 40 \\ 30 & 14 & -10 \\ -6 & 56 & 2 \end{pmatrix}$$

Find $|M|$ as $(2,0,6) \cdot (27, -56, 40) = 294$. Transpose C and scale each element by $1 / |M|$ to obtain

$$M^{-1} = \frac{1}{294}\begin{pmatrix} 27 & -56 & 40 \\ 30 & 14 & -10 \\ -6 & 56 & 2 \end{pmatrix}$$

Check this by multiplying out MM^{-1} and $M^{-1}M$.

The inverse is often used to solve a **set of linear equations**:

$$N\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

where an n by n matrix N is given, along with the column vector **b**, and it is necessary to find the vector **x** that causes all n of the equations to be satisfied simultaneously. If N is nonsingular, the solution may be found as

$\mathbf{x} = N^{-1}\mathbf{b}$

There are numerical techniques for solving such a system of equations that are faster and more numerically stable than computing $N^{-1}\mathbf{b}$ directly.

Note: Although the use of column vectors is prevalent in graphics, in certain fields it is more common to use row vectors and to write this same set of equations as

$(x_1, x_2, \ldots, x_n)\, M = (b_1, b_2, \ldots, b_n)$

It is not difficult to show that this is the same set of equations as the previous ones, when $M = N^T$, and that the solution is given by $\mathbf{x} = \mathbf{b}M^{-1}$.

Orthogonal Matrices.
For some transformations such as rotations (see Chapter 5), the associated matrix has an inverse that is particularly easy to find. A matrix M is called **orthogonal** if simply transposing it produces its inverse: $M^T = M^{-1}$. Therefore $MM^T = I$. If M is orthogonal, $MM^T = I$ implies that each of its rows is a unit length vector and that the rows are mutually orthogonal. The same is true for its columns (why?). For instance, if M is 3 by 3, partition it into three rows as follows:

$$M = \begin{pmatrix} \mathbf{a} \\ \hline \mathbf{b} \\ \hline \mathbf{c} \end{pmatrix}$$

Then the 3-tuples **a**, **b**, and **c** are each of unit length, and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} = 0$.

A Useful Identity for Cross Products.
When studying normal vectors to surfaces it is necessary to work with the cross product of two transformed 3D vectors, as in $(M\mathbf{a}) \times (M\mathbf{b})$, where **a** and **b** are 3D vectors, and *M* is a 3 by 3 matrix. The question is how this cross product is related to the cross product $\mathbf{a} \times \mathbf{b}$ of **a** and **b** alone. The answer is:

$$(M\mathbf{a}) \times (M\mathbf{b}) = (\det M) M^{-T} (\mathbf{a} \times \mathbf{b})$$

so $\mathbf{a} \times \mathbf{b}$ is scaled by the determinant of *M*, and multiplied by the inverse transpose of *M*. To establish this result the following steps may prove helpful. (Can you find a more immediate derivation?)

Denote the rows of *M* by the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$.

First show that $(M\mathbf{a}) \times (M\mathbf{b}) = \begin{pmatrix} \mathbf{r}_2 \times \mathbf{r}_3 \\ \hline \mathbf{r}_3 \times \mathbf{r}_1 \\ \hline \mathbf{r}_1 \times \mathbf{r}_2 \end{pmatrix} (\mathbf{a} \times \mathbf{b})$. Then show that

$$M^{-T} (M\mathbf{a}) \times (M\mathbf{b}) = \begin{pmatrix} \mathbf{r}_2 \times \mathbf{r}_3 \\ \hline \mathbf{r}_3 \times \mathbf{r}_1 \\ \hline \mathbf{r}_1 \times \mathbf{r}_2 \end{pmatrix} \left(\mathbf{r}_1^T \middle| \; \mathbf{r}_2^T \middle| \; \mathbf{r}_3^T \right) (\mathbf{a} \times \mathbf{b})$$

finally, show that the product of the first two matrices on the right hand side is a diagonal matrix, with each diagonal term equal to the determinant of *M*. (Hint: use properties of the triple scalar product discussed below, such as $\mathbf{a} \cdot \mathbf{c} \times \mathbf{a} = 0$.)

A2.2. Some Properties of Vectors and their Operations.

A2.2.1. The Perp of a vector, and the perp dot product.

The perp and perp dot product apply only to two dimensional vectors.

A). The "perp" of a vector. If vector **a** is given by $\mathbf{a} = (a_x, a_y)$, the **counterclockwise** perpendicular, or "perp" of **a**, denoted by $\mathbf{a}^\perp$, is given by $\mathbf{a}^\perp = (-a_y, a_x)$.

a). Vector **a** and $\mathbf{a}^\perp$ have the same length: $|\mathbf{a}| = |\mathbf{a}^\perp|$.

b). Linearity: $(\mathbf{a} + \mathbf{b})^\perp = \mathbf{a}^\perp + \mathbf{b}^\perp$ and $(A\mathbf{a})^\perp = A\mathbf{a}^\perp$ for any scalar *A*;

c). Two perp's make a negation: $\mathbf{a}^{\perp\perp} = (\mathbf{a}^\perp)^\perp = -\mathbf{a}$

B). The perp dot product $\mathbf{a}^\perp \cdot \mathbf{b}$.

a). The value of the perp dot product: $\mathbf{a}^\perp \cdot \mathbf{b} = a_x b_y - a_y b_x$.

b). $\mathbf{a}^\perp \cdot \mathbf{a} = 0$, ($\mathbf{a}^\perp$ is perpendicular to **a**)

c). $|\mathbf{a}^\perp|^2 = |\mathbf{a}|^2$. ($\mathbf{a}^\perp$ and **a** have the same length) d). $\mathbf{a}^{\perp} \cdot \mathbf{b} = - \mathbf{b}^{\perp} \cdot \mathbf{a}$, (antisymmetric)

e). $\mathbf{a}^{\perp} \cdot \mathbf{b}$ can be written as the determinant:

$$\mathbf{a}^{\perp} \cdot \mathbf{b} = \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

f). $(\mathbf{a}^{\perp} \cdot \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$.

g). If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ then $\mathbf{a}^{\perp} \cdot \mathbf{b} = \mathbf{b}^{\perp} \cdot \mathbf{c} = \mathbf{c}^{\perp} \cdot \mathbf{a}$.

h). $\mathbf{a}^{\perp} \cdot \mathbf{b} > 0$ if and only if there is a CCW turn from **a** to **b**.

i). $\mathbf{a}^{\perp} \cdot \mathbf{b} = 0$ if **b** is parallel or anti-parallel to **a**

j). $|\mathbf{a}^{\perp} \cdot \mathbf{b}|$ is the area of the parallelogram determined by vectors **a** and **b**.

A2.2.2. The Scalar Triple Product

For vectors **a**, **b**, and **c** of 3 dimensions, a very useful quantity combines the cross product with the dot product. Given three vectors, **a**, **b**, and **c**, create the scalar S defined by

$$S = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - b_x c_z) + a_z(b_x c_y - b_y c_x)$$

This can also be written conveniently as the determinant:

$$S = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Interchanging the rows of a determinant causes only a change in sign, and so interchanging twice produces no change at all. Hence a cyclic permutation in the vectors has no effect on the value of S, and it has the following three equivalent forms:

$$S = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \bullet (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})$$

The scalar triple product has a simple geometric interpretation. (It plays the same role in 3D as the perp dot product $\mathbf{b}^{\perp} \cdot \mathbf{c}$ plays in 2D.

• Its *magnitude* $|S|$ is the volume of the parallelepiped[1] formed by the vectors **a**, **b**, and **c** all bound to the same point.

• The *sign* of the triple scalar product follows that of $\cos(\phi)$: positive if $|\phi| < 90°$ and negative if $|\phi| > 90°$. (Question: If we express **a**, **b**, and **c** instead in a right handed coordinate system, does S change?).

Note that if the three vectors lie in the same plane, the scalar triple product will be zero, as the volume of the parallelepiped then degenerates to zero. Suppose that none of **a**, **b**, or **c** is the zero vector. Then the

[1]**parallelepiped** (pronounced with the syllable "ep" stressed, as in 'epithet' and 'epicycloid').

scalar triple product **a** • (**b** X **c**) = 0 if, and only if, the three vectors are coplanar. (Corollary: The three vectors are coplanar if any two of them are parallel.) This property can be used to determine how nearly planar a polygon is.

The Intersection of Three Planes. Two planes intersect in a line, and a third plane intersects this line at a single point. The scalar triple product provides a closed-form expression for this point. If the planes are given by $\mathbf{n}_i \cdot \mathbf{r} = D_i$, for i = 1, 2, 3, their point of intersection is given (provided that the denominator is not zero) by:

$$\mathbf{r} = \frac{D_1(\mathbf{n}_2 \times \mathbf{n}_3) + D_2(\mathbf{n}_3 \times \mathbf{n}_1) + D_3(\mathbf{n}_1 \times \mathbf{n}_2)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)})$$

This can be checked by seeing that **r** lies in each of the three planes: Substitute the expression for each plane in this formula, and use the properties of the triple scalar product to show that an equality results.

A2.2.4. The Triple Vector Product and products of four vectors.

The triple vector product (TVP) of three 3D vectors, **a**, **b**, and **c**, is given by TVP = $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. It often arises during pencil-and-paper calculations involving cross products. It can be written as the difference of the two scaled vectors: TVP = $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ [Faux79].

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"Products" of four vectors. For any four 3D vectors **a**, **b**, **c**, and **d** the following is true:
$(\mathbf{a} \times \mathbf{b})\bullet(\mathbf{c} \times \mathbf{d}) = (\mathbf{a}\bullet\mathbf{c})(\mathbf{b}\bullet\mathbf{d}) - (\mathbf{a}\bullet\mathbf{d})(\mathbf{b}\bullet\mathbf{c})$.

A2.3. The Arithmetic of Complex Numbers.

It is not essential to bring complex numbers into play when studying geometric methods in computer graphics. Complex numbers and their manipulations lend considerable insight into various facts, however, making a study of them well worth while. This appendix collects the elementary facts of complex arithmetic in one place, as a refresher for readers who have some familiarity with them.

A complex number z such as z = 3 + 4i has two parts. Its **real** part, denoted $Re(z)$, is equal to 3, and its so-called **imaginary**, denoted $Im(z)$, is 4. The quantity i, defined by $i^2 = -1$, is usually written $i = \sqrt{-1}$. There is nothing either complex or imaginary about these objects: they are simply defined according to a set of rules by which they operate. In performing arithmetic the usual operations apply:

• addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$ (A.2.1)
• multiplication: $(a + bi) * (c + di) = (ac - bd) + (ac + cd)i$

where the term bdi^2 has been replaced by $-bd$ according to the rule $i^2 = -1$. For instance, $(3 + 2i) + (1 + i) = 4 + 3i$, and $(3 + 2i) * (1 + i) = 1 + 5i$.

It is in the correspondence between complex numbers and points in the plane that complex numbers and their operations take on a rich geometric character. The complex number $x + yi$ is associated with the point (x, y) in the usual rectangular coordinate system. The x-coordinate is the real part of the number and the y-coordinate is the imaginary part. Thus 3 + 4i corresponds to (3, 4). Any complex number may be 'plotted' in the plane. Such a representation is called an **Argand diagram**[2]. Figure A2.2.1a shows the

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[2] Named after Jean Robert Argand, a Swiss bookeeper, who described it in 1806. The Norwegian surveyor Casper Wessel had actually described it nine years earlier, and Gauss used it at about the same time.

complex number 3 + 4*i* plotted as (3,4). The usual *x*-axis is often called the 'real' axis, and the *y*-axis the 'imaginary' axis, to reinforce the association.

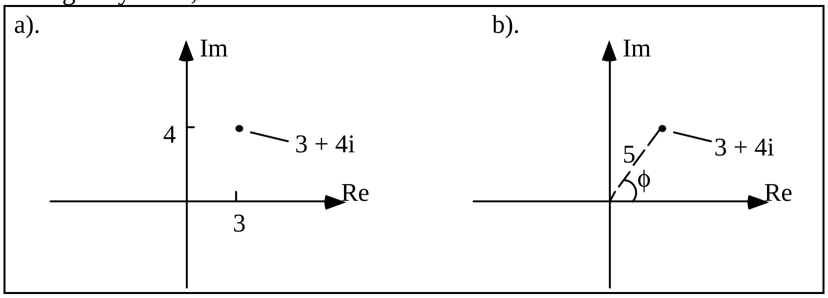

Figure A2.2.1. The Argand Diagram.

Just as the point (3, 4) is at distance $5 = \sqrt{3^2 + 4^2}$ from the origin, a complex number $c = a + bi$ is said to have **magnitude** or **modulus**, denoted by $|c|$ and given by

$$|c| = \sqrt{a^2 + b^2} \qquad \text{(A.2.2)}$$

Not surprisingly the **angle** or **argument** of $c = a + bi$ is the angle ϕ shown in Figure A2.1b. The argument of z is often denoted $Arg(z)$, so $Arg(z) = \phi$. Thus the real part of c is $|c|\ cos\ \phi$ and the imaginary part is $|c|\ sin\ \phi$, so we can write the 'polar' form:

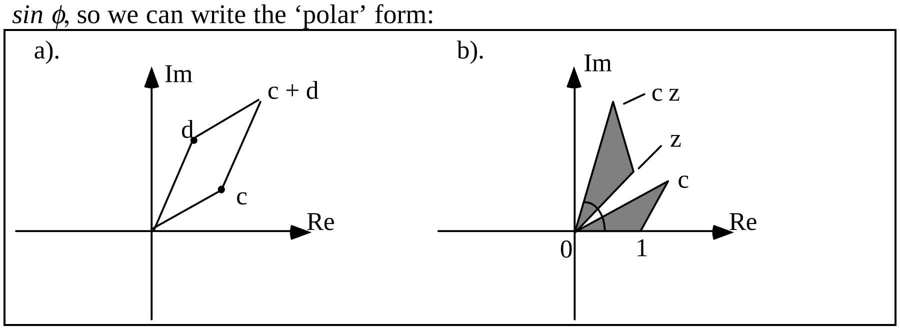

Figure A2.2.2. Adding and multiplying complex numbers.

$$c = |c|\ cos\ \phi + i\ |c|\ sin\ \phi \qquad \text{(A.2.3)}$$

More generally suppose c is given by Equation 5.43 and that z has polar form $z = |z|(\cos\ \theta + i\ sin\ \theta)$ (i.e. magnitude $|z|$ and argument θ). Now multiply them and simplify the result to get:

$$\begin{aligned} c * z &= |c|\ (cos\ \phi + i\ sin\ \phi)\ |z|\ (cos\ \theta + i\ sin\ \theta) \\ &= |c|\ |z|\ (cos\ (\phi + \theta) + i\ sin\ (\phi + \theta)) \end{aligned} \qquad \text{(A2.2.4)}$$

We conclude:
• the magnitude of the product of two complex numbers is the product of their magnitudes;
• the angle of the product of two complex numbers is the sum of their angles.

This is illustrated in Figure A2.2b. Note that the triangle formed by 0, 1, and c is similar to that formed by 0, z, and cz, so multiplying by a complex number converts a triangle into a similar triangle.

Letting $z = c$ in Equation A2.34, we obtain $z^2 = |z|^2\ (cos\ 2\phi + i\ sin\ 2\phi)$, which generalizes immediately to an expression for z^n (which one?) Letting $|z| = 1$ we get DeMoivre's reknowned formula:

$$(cos\ \phi + i\ sin\ \phi)^n = cos\ (n\phi) + i\ sin(n\phi) \qquad \text{(A2.2.5)}$$

We examine the function $\cos\phi + i\sin\phi$ more closely. Call it $f(\phi)$. The formula says: $f^n(\phi) = f(n\phi)$: raising the function to a power n is the same as multiplying its argument by n. This is highly suggestive of an exponential function, and in fact can be proven rigorously to be so. It produces **Euler's formula**:

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{A2.2.6}$$

(Proof: both sides have the same infinite series expansion.) As special cases, note that $e^{i0} = 1$, $e^{i\pi/2} = i$, and $e^{i\pi} = -1$. (This last relates in a remarkable way the four fundamental mathematical values: e, i, π, and 1.) This provides us with an alternative and very compact polar form for a complex number c having magnitude $|c|$ and angle ϕ:

$$c = |c|\, e^{i\phi} \tag{A2.2.7}$$

For example, the n vertices of an n-gon of radius R are given by the n complex numbers p_k:

$$p_k = R\, e^{i2\pi k/n} \qquad k = 1, 2, .., n \tag{A2.2.8}$$

Each complex number z also has a **conjugate**, denoted z^*. If $z = x + iy$ then by definition $z^* = x - iy$. Thus $|z^*| = |z|$ and $Arg(z^*) = -Arg(z)$. Taking the conjugate is equivalent to areflection about the x-axis. (What are the magnitude and argument of $(z^*)^n$?)

• The square root $\sqrt{z}$ of a complex number z.

If the complex number z has the polar form $z = |z|\, e^{j\phi}$ then clearly the square root of z is :

$$\sqrt{z} = \sqrt{|z|}\; e^{i\phi/2}.$$

Thus taking the square root takes the square root of the magnitude and halves the argument of z. This can also be written without recourse to the polar form. If $z = x + iy$ then:

$$\sqrt{z} = a + ib \qquad \text{if } y \bullet 0 \tag{A2.2.9}$$
$$\sqrt{z} = -a + ib \qquad \text{if } y < 0$$

where

$$a = \sqrt{\frac{|z|+x}{2}} \qquad b = \sqrt{\frac{|z|-x}{2}} \tag{A2.2.10}$$

Check: Square $a + ib$ and $-a + ib$ directly and work out the algebra, to see that the result is z itself.

Practice Exercises.

A2.2.1. The division operation. If z ands w are complex numbers, show that

$$\frac{z}{w} = \frac{|z|}{|w|} e^{i(Arg(z) - Arg(w))} = \frac{zw^*}{ww^*} = \frac{zw^*}{|w|^2} \tag{A2.2.11}$$

A2.2.2. A ratio that always has unit magnitude.

Show that $(a+ib) / (a - ib)$ has magnitude 1 for any a and b.

A2.2.3. When must four complex numbers lie on a circle?

Show that $Arg((z_3 - z_1) / (z_2 - z_1)) = Arg((z_4 - z_1) / (z_4 - z_2))$ if and only if $z_1,..,z_4$ lie on a circle or straight line, if and only if $[(z_3 - z_1) / (z_3 - z_2)] / [(z_4 - z_1) / (z_4 - z_2)]$ is real.

A2.4. Spherical Coordinates and Direction Cosines.

We review the notion of spherical coordinates, and summarize how you convert back and forth from spherical coordinates to Cartesian coordinates.

Figure 9.8.1 shows how a point *U* is defined in spherical coordinates. *R* is the radial distance of *U* from the origin. ϕ is the angle that *U* makes with the *xy*-plane, known as the **latitude** of point *U*. θ is the **azimuth** of *U*: the angle between the *xz*-plane and the plane through *U* and the *z*-axis. ϕ lies in the interval $-\pi/2 \le \phi < \pi/2$, and θ lies in $0 \le \theta < 2\pi$.

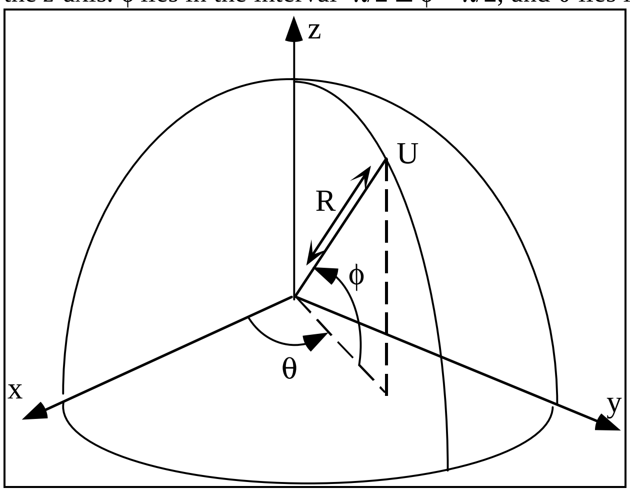

Figure 9.8.1. Spherical Coordinates.

Using simple trigonometry, it is straightforward to work out the relationships between these quantities and the Cartesian coordinates (u_x, u_y, u_z) for *U*. They are

$$
\begin{aligned}
u_x &= R\cos(\phi)\cos(\theta) \\
u_y &= R\cos(\phi)\sin(\theta) \\
u_z &= R\sin(\phi)
\end{aligned}
\tag{9.8.1}
$$

One can also invert these relations to express (R, ϕ, θ) in terms of (u_x, u_y, u_z):

$$R = \sqrt{u_x^{\ 2} + u_y^2 + u_z^2}$$

$$\phi = \sin^{-1}(\frac{u_z}{R}) \tag{9.8.2}$$

$$\theta = \arctan(u_y, u_x)$$

The function arctan(,) is the two-argument form of the arctangent, defined as

$$\arctan(y,x)=\begin{cases}\tan^{-1}(y/x) & \text{if } x>0\\ \pi+\tan^{-1}(y/x) & \text{if } x<0\\ \pi/2 & \text{if } x=0 \text{ and } y>0\\ -\pi/2 & \text{if } x=0 \text{ and } y<0\end{cases} \tag{9.8.3}$$

It can distinguish between the case where both x and y are positive and the case where both of them are negative, unlike the usual form $\tan^{-1}(\frac{y}{x})$, which always produces angles between -•/2 and •/ 2.

Example 9.8.1. Suppose that point U is at distance 2 from the origin, is 60° up from the xy-plane, and is along the negative x-axis. Hence U is in the xz-plane. Then U is expressed in spherical coordinates as (2, 60°, 180°). Using Equation 9.8.1 to compute U in Cartesian coordinates, we obtain U = (-1, 0, 1.732).

• **Direction Cosines.** The direction of point U in the preceding example is given in terms of two angles, the azimuth and the latitude. Directions are often specified in an alternative useful way through direction cosines. The direction cosines of a line through the origin are the cosines of the three angles it makes with the x-, y-, and z-axes, respectively.

Recall that the cosine of the angle between two unit vectors is given by their dot product. Using the given point U, form the position vector (u_x, u_y, u_z). From the preceding discussion we see that its length is R, so it must be normalized to the unit length vector **m** = (u_x / R, u_y / R, u_z / R). Then the cosine of the angle it makes with the x-axis is given by the dot product **m** · **i** = u_x / R, which is simply the first component of **m**. Similarly, the second and third components of **m** are the second and third direction cosines, respectively. Calling the angles made with the x-, y-, and z-axes by α, β, and γ, respectively, the three direction cosines for the line from 0 to U are therefore

$$\begin{aligned}\cos(\alpha)&=\frac{u_x}{R}\\ \cos(\beta)&=\frac{u_y}{R}\\ \cos(\gamma)&=\frac{u_z}{R}\end{aligned} \tag{9.8.4}$$

Note that the three direction cosines are related, since the sum of their squares is always 1.

Practice Exercises.
9.8.1. Convert the point (x, y, z) = (2, 4, -3) to spherical coordinates.
9.8.2. Convert the point (r,ϕ, θ) = (5, 35°, -67°) to rectangular coordinates.
9.8.3. Find the direction cosines of the vector **n**, where
i). **n** = (1, 1, 1);
ii). **n** = (2, 3, 4).